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On Transforming a Tchebyshev-System into a Markov-System

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1. INTRODUCTION

Let $f_1,..., f_n$ be real-valued functions defined on a set M. $f_1,..., f_n$ form a T-system (Tchebyshev-system) iff every nontrivial linear combination of them has at most $n - 1$ zeros. $f_1,..., f_n$ form a Markov-system (called a "complete T-system" by Karlin and Studden [2]) iff $f_1,..., f_i$ form a T-system for $i = 1,..., n$. The linear hull of a T-system will be called a T-space. An n-dimensional T-space R has a basis that is a Markov-system iff there exist i-dimensional T-spaces U_i, $i = 1,..., n$, with $U_1 \subset U_2 \subset \cdots \subset U_n = R$.

Rutman [5] quotes the following theorem of Krein: If M is an open interval and $R \subset C(M)$ is an n-dimensional T-space, then R has a Markov-basis. We shall show that this proposition holds if only the following conditions are fulfilled:

(1) M is totally ordered, contains no smallest or greatest element, and for every two distinct elements of M there is an element between them.

(2) For every function f in the T-space there are at most n points $t_1,..., t_n$ in M with $t_1 < \cdots < t_n$ such that f changes sign in each of them.

In Section 2 basic properties of T-spaces are listed, and a generalization of a theorem of Nemeth [4] is proved.

Restriction to totally ordered domains in Section 3 allows the definition of "alternations"; also some results about T-spaces with certain alternation properties are derived.

After the proof of our main theorem in Section 4 we list examples of T-spaces that do not have a Markov-basis. It should be mentioned that no such example is known for $n = 4, 6, 8,...$; for $n = 5, 7, 9,...$ the only examples known consist of periodic functions.

In Section 5 we give a short proof of Mairhuber's theorem which simplifies the proof given by Schoenberg and Yang [6]. Most proofs in this paper are based on alternation properties of functions in a T-space and seem to be simpler than proofs using determinants.

2. Definitions and Basic Properties

Let M be a nonempty set and $\mathbb{R}^M$ the linear space of real-valued functions defined on M. An n-dimensional linear subspace R of $\mathbb{R}^M$ is called a T-space (Tchebyshev-space) iff every $f \in R$ with n or more zeros vanishes identically.

Let R^* the space of linear functionals on R and $M^* \subset R^*$ the set of all linear functionals t^* for which there is a $t \in M$ with $t^*(f) = f(t)$ for all $f \in R$, i.e., M^* is the set of the point functionals on R.

For a set $N \subset M$ we define the projection $E_N{}^M$: $R^M \to R^N$ by $E_N{}^M(f) = f|_N$ for all $f \in \mathbb{R}^M$.

We omit the proof of the following lemma, which is fairly obvious.

Lemma 1. *Let $R \subset \mathbb{R}^M$ be an n-dimensional linear space, $n \geqslant 1$. Then the following properties are equivalent:*

(a) *R is an n-dimensional T-space on M.*

(b) *For every subset N of M with n elements $E_N{}^M(R)$ is an n-dimensional T-space on N.*

(c) *If $t_1{}^*,..., t_n{}^* \in M^*$ are pairwise different, they are linearly independent.*

(d) *For every basis $f_1 ,..., f_n$ of R and every set $t_1 ,..., t_n \in M$ of pairwise distinct points we have* $\det(f_i(t_j))_{n,n} \neq 0$.

(e) *If $t_1 ,..., t_n \in M$ are pairwise distinct, and $\alpha_1 ,..., \alpha_n \in \mathbb{R}$, there is exactly one $f \in R$ with $f(t_i) = \alpha_i$ for $i = 1,..., n$.*

For an n-dimensional T-space on a set M we denote by Z_k the set of all $f \in R\backslash\{0\}$ with at least k zeros, $k = 1,..., n$.

Theorem 1. *Let R be an n-dimensional T-space on a set M, and $n \geqslant k \geqslant 0$. Then the following statements are equivalent:*

(a) *R contains an $(n-k)$-dimensional T-space on M.*

(b) *R^* contains a k-dimensional linear subspace W such that for each $f \in Z_{n-k}$ there is a $t^* \in W$ with $t^*(f) \neq 0$.*

Proof. (a) $\Rightarrow$ (b). Let $U \subset R$ be an $(n-k)$-dimensional T-space on M and $W := U^{\perp}$. W is a k-dimensional linear space. As we have $Z_{n-k} \cap U = \varnothing$, it follows that for every $f \in Z_{n-k}$ there is a $t^* \in W$ with $t^*(f) \neq 0$.

(b) $\Rightarrow$ (a). Let $U := W^{\perp} = \{f \in R \mid t^*(f) = 0 \text{ for all } t^* \in W\}$. U is an $(n-k)$-dimensional T-space on M because of $Z_{n-k} \cap U = \varnothing$.

As a special case of Theorem 1 we get a result of Nemeth [4].

COROLLARY 1. *Let R be an n-dimensional T-space on a set M, and $n \geqslant 1$. Then the following statements are equivalent:*

(a) *R contains an $(n-1)$-dimensional T-space on M.*

(b) *There is a point $u \notin M$ and an n-dimensional T-space S on $M \cup \{u\}$ such that $E_M^{M \cup \{u\}}(S) = R$.*

Proof. For $n = 1$(a) and (b) are always true. Let $n \geqslant 2$. By Theorem 1(a) is equivalent to the existence of a one-dimensional subspace $W \subset R^*$ such that for all $f \in Z_{n-1}$ there is a $t^* \in W$ with $t^*(f) \neq 0$. Hence, (a) is equivalent to the existence of a $t \in R$ such that for all $f \in Z_{n-1} t^*(f) \neq 0$. As there is a one-to-one correspondence between M and M^* for $n \geqslant 2$, and $t^* \notin M^*$, the equivalence with (b) follows by setting $f(u) = t^*(f)$.

Remark. It is easy to see that under the hypotheses of Corollary 1, if $f_1, ..., f_n$ is a basis of R, and $\phi: M \to \mathbb{R}^n$ is the mapping defined by $\phi(t) = (f_1(t), ..., f_n(t))$, (a) and (b) are equivalent to the following statement:

(c) There is an $x \in \mathbb{R}^n$ such that every $(n-1)$-dimensional hyperplane through x and 0 intersects $\phi(M)$ in at most $n - 2$ points (see Hadeler [1]).

3. Totally Ordered Domains

In the following we assume M to be totally ordered by "$<$".

DEFINITION. Let $f \in R^M$, and $t_1, ..., t_k \in M$ with $t_1 < \cdots < t_k$.

(a) $t_1, ..., t_k$ form a strong alternation of f of length k iff either $(-1)^i f(t_i) > 0$ for $i = 1, ..., k$ or $(-1)^i f(t_i) < 0$ for $i = 1, ..., k$.

(b) $t_1, ..., t_k$ form a weak alternation of f of length k iff either $(-1)^i f(t_i) \geqslant 0$ for $i = 1, ..., k$ or $(-1)^i f(t_i) \leqslant 0$ for $i = 1, ..., k$.

LEMMA 2. *Let M be totally ordered, and $R \subset \mathbb{R}^M$ an n-dimensional linear space, $n \geqslant 1$. Then the following statements are equivalent:*

(a) *If $f_1, ..., f_n$ is a basis of R, then either $\det(f_i(t_j))_{n,n} > 0$ for all $t_1, ..., t_n \in M$ with $t_1 < \cdots < t_n$, or $\det(f_i(t_j))_{n,n} < 0$ for all $t_1, ..., t_n \in M$ with $t_1 < \cdots < t_n$.*

(b) *R is a T-space on M, and for arbitrary $f \in R$ every strong alternation of f has length less or equal n.*

(c) *R is a T-space on M, and for arbitrary $f \in R \backslash \{0\}$ every weak alternation of f has a length less or equal n.*

Proof. (c) $\Rightarrow$ (a). Let $f_1, ..., f_n$ be a basis of R. By Lemma 1 we have $\det(f_i(t_j))_{n,n} \neq 0$ for all $t_1, ..., t_n \in M$ with $t_1 < \cdots < t_n$. Suppose that there are points $u_1, ..., u_n, v_1, ..., v_n \in M$ with $u_1 < \cdots < u_n$, $v_1 < \cdots < v_n$ and $\operatorname{sign} \det(f_i(u_j))_{n,n} = -\operatorname{sign} \det(f_i(v_j))_{n,n} = 1$. Let $W = \{u_1, ..., u_n\} \cup \{v_1, ..., v_n\}$ and $w_1, ..., w_n$ the first n points of W with respect to ordering. Successively replacing u_1 by w_1, then u_2 by w_2, and so on until replacing u_n by w_n, and then w_n by v_n, w_{n-1} by v_{n-1}, and so on until finally replacing w_1 by v_1, we obtain a sequence of $2n - 1$ sets each of which contains n points and differs from its neighbors in at most one point. Considering the sign of the determinants corresponding to each element of the sequence, it becomes apparent that without loss of generality we may assume that $u_1, ..., u_n$ and $v_1, ..., v_n$ differ by one point only, for instance $u_i < v_i$ for a fixed i and $u_k = v_k$ for $k = 1, ..., i - 1, i + 1, ..., n$. Let $g \in R$ be defined by

$$g(t) := \det \begin{pmatrix} & f_1 \cdots f_n & \\ u_1 \cdots u_{i-1}, & t, & u_{i+1} \cdots u_n \end{pmatrix}$$

$$= \begin{vmatrix} f_1(u_1) & \cdots & f_1(u_{i-1}) & f_1(t) & f_1(u_{i+1}) & \cdots & f_1(u_n) \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ f_n(u_1) & \cdots & f_n(u_{i-1}) & f_n(t) & f_n(u_{i+1}) & \cdots & f_n(u_n) \end{vmatrix}$$

g has zeros in $u_1, ..., u_{i-1}, u_{i+1}, ..., u_n$ and opposite sign in u_i and v_i. So $u_1, ..., u_i, v_i, u_{i+1}, ..., u_n$ is a weak alternation of g of length $n + 1$.

(a) $\Rightarrow$ (b). Obviously R is a T-space. Suppose that there are $f \in R$ and $t_1, ..., t_{n+1} \in M$ with $t_1 < \cdots < t_{n+1}$ and $(-1)^i f(t_i) < 0$ for $i = 1, ..., n + 1$. Let $f_1, ..., f_n$ be a basis of R. As $\det(f_i(u_j))_{n,n}$ has constant sign for all $u_1, ..., u_n \in M$ with $u_1 < \cdots < u_n$, we have

$$\det \begin{pmatrix} f & f_1 & \cdots & f_n \\ & t_1 \cdots t_{n+1} & & \end{pmatrix} = \begin{vmatrix} f(t_1) & \cdots & f(t_{n+1}) \\ f_1(t_1) & \cdots & f_1(t_{n+1}) \\ \vdots & & \vdots \\ f_n(t_1) & \cdots & f_n(t_{n+1}) \end{vmatrix}$$

$$= \sum_{1}^{n+1} |f(t_i)| \det \begin{pmatrix} & f_1 \cdots f_n & \\ t_1 \cdots t_{i-1}, & t_{i+1} \cdots t_{n+1} \end{pmatrix} \neq 0,$$

and $f, f_1, ..., f_n$ are linearly independent in contradiction to $\dim R = n$.

(b) $\Rightarrow$ (c). Suppose that there are $f \in R \backslash \{0\}$ and $t_1, ..., t_{n+1} \in M$ with

$t_1 < \cdots < t_{n+1}$ and $(-1)^i f(t_i) \geqslant 0$ for $i = 1,..., n + 1$. As R is a T-space, there is a k, $1 \leqslant k \leqslant n + 1$, with $(-1)^k f(t_k) > 0$.

Let $g \in R$ with $g(t_i) = (-1)^i$ for $i = 1,..., k - 1, k + 1,..., n + 1$. For all $\alpha > 0$ we have $(-1)^i [(f + \alpha g)(t_i)] > 0$ for $i = 1,..., k - 1, k + 1,..., n + 1$, and there is a $\beta > 0$ with $(-1)^k [(f + \beta g)(t_k)] > 0$. So $t_1 ,..., t_{n+1}$ is a strong alternation of $f + \beta g$ of length $n + 1$.

DEFINITION. Let M be totally ordered, and let $R \subset \mathbb{R}^M$ an n-dimensional linear space. If R has one (and, hence, all) of the properties (a), (b), or (c), R is called an oriented T-space.

LEMMA 3. *Let M be totally ordered and $R \subset \mathbb{R}^M$ an n-dimensional oriented T-space on M, and $f \in Z_{n-1}$ with zeros $t_1 < \cdots < t_{n-1}$.*

(a) *All point sets $s_1 ,..., s_n \in M$ with $s_1 < t_1 \leqslant s_2 < t_2 \leqslant \cdots < t_{n-1} \leqslant s_n$ or $s_1 \leqslant t_1 < s_2 \leqslant t_2 < \cdots \leqslant t_{n-1} < s_n$ form weak alternations of f of length n.*

(b) *All point sets $s_1 ,..., s_n \in M$ with $s_1 < t_1 < s_2 < t_2 < \cdots < t_{n-1} < s_n$ form strong alternations of f of length n.*

Proof. (a) Without loss of generality assume $s_1 < t_1 \leqslant s_2 < t_2 \leqslant \cdots < t_{n-1} \leqslant s_n$ and $f(s_1) > 0$. If there were an i with $(-1)^i f(s_i) > 0$, the points $s_1 , t_1 ,..., t_{i-1} , s_i , t_i ,..., t_{n-1}$ would form a strong alternation of f of length $n + 1$.

(b) By (a) we know that all point sets $s_1 ,..., s_n \in M$ with $s_1 < t_1 < s_2 < t_2 < \cdots < t_{n-1} < s_n$ form weak alternations of f of length n. If we had $f(s_j) = 0$ for some j, f had n zeros in contradiction to $f \in Z_{n-1}$.

DEFINITION. Let R be an n-dimensional oriented T-space on a totally ordered set M. By A we denote the set of all $f \in R$ which have a strong alternation of length n.

DEFINITION. A totally ordered set M has property (D) if for all $t \in M$ there are points $u, v \in M$ with $u < t < v$, and for all $x, y \in M$ with $x < y$ there is a $z \in M$ with $x < z < y$.

As an immediate consequence of Lemma 3 we get the following lemma.

LEMMA 4. *If M is a totally ordered set with property (D) and R is an n-dimensional oriented T-space on M, Z_{n-1} is a subset of A.*

4. $(n-1)$-Dimensional Subspaces

In a finite-dimensional linear space all norms generate the same topology. In the following let the n-dimensional space R be endowed with this topology.

Lemma 5. *Let M be totally ordered and R an n-dimensional T-space on M. Then for every $f \in A$ (A being the set of all functions in R with a strong alternation of length n) there is an open neighborhood $N(f)$ of f with $N(f) \subset A$.*

Proof. Let $f \in A$ have a strong alternation in $t_1 < \cdots < t_n$, and define $\alpha := \min |f(t_i)|$. If $\| \ \|$ denotes the maximum norm on $\{t_1, ..., t_n\}$ in R, then for every $g \in R$ with $\| g \| < \alpha$ the points $t_1, ..., t_n$ form a strong alternation of $f - g$, and, thus, $f - g \in A$.

For the proof of our main theorem we need the following result from linear algebra.

Lemma 6. *Let R be an n-dimensional linear space, I a totally ordered index set, and $F = \{B_i \subset R \mid i \in I\}$ an antitone family of closed sets with $B_i \subset B_j$ for all $i, j \in I$ with $i > j$, and for every i let B_i contain a linear space U_i of dimension k. Then $B := \bigcap \{B_i \mid i \in I\}$ contains a k-dimensional linear space.*

Proof. The statement is trivial for finite I. If I is infinite, let R be endowed with an inner product, and for every $i \in I$, let $e_i^1, ..., e_i^k$ be an orthonormal basis of U_i. Now consider the k-fold Cartesian product $R \times \cdots \times R$. The set $N := \{(e_i^1, ..., e_i^k) \in R \times \cdots \times R \mid i \in I\}$ is bounded, and we have $(e_i^1, ..., e_i^k) \in B_j \times \cdots \times B_j$ for all $i, j \in I$ with $i > j$. Since

$$S := \{(x_1, ..., x_k) \in R \times \cdots \times R \mid \| x_1 \| = \cdots = \| x_k \| = 1\}$$

is compact, N has a cluster point $(e^1, ..., e^k) \in S$, and since $B_i \times \cdots \times B_i$ is closed for every i, we have $(e^1, ..., e^k) \in B_i \times \cdots \times B_i$ for all $i \in I$. So we have $(e^1, ..., e^k) \in B \times \cdots \times B$ or $e^1, ..., e^k \in B$. It is easy to see that $e^1, ..., e^k$ are pairwise orthogonal and that $\operatorname{span}\{e^1, ..., e^k\}$ lies in B.

Theorem 2. *Let M be totally ordered and have property (D), and R an n-dimensional oriented T-space on M, $n \geqslant 1$. Then R contains a $(n-1)$-dimensional oriented T-subspace on M.*

Proof. For $t \in M$ we define

$$\begin{aligned} M_t &:= \{u \in M \mid u < t\}, \\ A_t &:= \{f \in R \mid f|_{M_t} \text{ has a strong alternation of length } n\}, \\ B_t &:= R \backslash A_t, \\ U_t &:= \{f \in R \mid f(t) = 0\}. \end{aligned}$$

Since $M_s \subset M_t$ for $s < t$ and $\bigcup\{M_t \mid t \in M\} = M$, we have $A_s \subset A_t$ for $s < t$ and $\bigcup\{A_t \mid t \in M\} = A$, and thus $B_s \subset B_t$ for $s > t$. By Lemma 5 the A_t are open; so the B_t are closed. If an $f \in U_t$ had a strong alternation of length n on M_t, f would have a weak alternation of length $n + 1$ on M in contradiction to the hypothesis that R is oriented. So we have $U_t \subset B_t$ for all $t \in M$.

As the U_t are $(n - 1)$-dimensional linear spaces, the hypotheses of Lemma 6 are fulfilled; thus, $B = R \backslash A$ contains an $(n - 1)$-dimensional space U, and $U \cap A = \varnothing$.

As R is oriented and M has property (D), by Lemma 4 we have $Z_{n-1} \subset A$, and so $U \cap Z_{n-1} = \varnothing$. Then U is an $(n - 1)$-dimensional oriented T-space on M, for no $f \in U \backslash \{0\}$ has more than $n - 2$ zeros or an alternation of length greater $n - 1$.

Repeated application of Theorem 2 yields the following corollary.

COROLLARY 2. *Let M be totally ordered and have property (D), R an n-dimensional oriented T-space on M, and $n \geqslant 1$. Then for $i = 1,\dots, n$ there exist i-dimensional oriented T-spaces with $U_1 \subset U_2 \subset \cdots \subset U_n = R$.*

We now give some examples of n-dimensional oriented T-spaces that do not contain T-subspaces of dimension $n - 1$.

(1) All T-spaces of continuous 2π-periodic functions on the half-open interval $[0, 2\pi)$ have odd dimension.

Proof. If R is such a T-space, let $f \in Z_{n-1}$ with zeros $t_1 < \cdots < t_{n-1}$, $0 < t_1$, $t_{n-1} < 2\pi$. From Lemma 3 we see that f changes sign in each of the t_i's. So for sufficiently small $\epsilon > 0$ we have

$$\operatorname{sign} f(t_1 - \epsilon) = \operatorname{sign} f(t_{n-1} + \epsilon) = (-1)^{n-1} \operatorname{sign} f(t_1 - \epsilon),$$

and n has to be odd.

(2) Let $f_1(t) = \sin t$, $f_2(t) = \cos t$ for $t \in M = [0, \pi)$. Then $R := \operatorname{span}\{f_1, f_2\}$ contains no one-dimensional T-subspace.

(3) Let $f_1(t) = 1$, $f_2(t) = t \sin t$, $f_3(t) = t \cos t$ for $t \in M = [0, \pi]$. Then $R := \operatorname{span}\{f_1, f_2, f_3\}$ is an oriented T-space on $[0, \pi]$ which contains no two-dimensional T-space.

Proof. (a) *R is a T-space.* Every $f \in R$ can be written $f(t) = \alpha + \beta t \sin(t - p)$ with suitable $\alpha, \beta \in \mathbb{R}$, $\beta \geqslant 0$, and a phase shift $p \in [0, 2\pi)$. If $\alpha = 0 \neq \beta$, f has exactly two zeros in $[0, \pi]$ because $\sin(t - p)$ has exactly one zero in $(0, \pi]$. If $\alpha \neq 0$, $f(t) = 0$ is equivalent to $1/t = (\beta/\alpha) \sin(t - p)$. As $1/t$ is positive and convex on $(0, \pi]$, and the positive part of $(\beta/\alpha) \sin(t - p)$ is concave on $(0, \pi]$, there can be at most two zeros of f in $(0, \pi]$.

(b) *R contains no two-dimensional T-space on M.* By Theorem 1 it is sufficient to show that for each $t^* \in R^*$ there is an $f \in Z_2$ with $t^*(f) = 0$. Let $t^* \in R^*$ with $t^*(f_i) = \alpha_i$, $i = 1, 2, 3$. For $\alpha_2 = 0$ the statement is true because of $f_2 \in Z_2$. For $\alpha_2 \neq 0$ we have $t^*(\alpha_3 f_2 - \alpha_2 f_3) = 0$, and $(\alpha_3 f_2 - \alpha_2 f_3)(t) = t(\alpha_3 \sin(t) - \alpha_2 \cos t)$ has a zero in 0 and another one in $(0, \pi]$.

Infinitely many examples of three-dimensional oriented T-spaces that contain no two-dimensional subspaces can be obtained by means of the remark at the end of Section 2: If f_1, f_2, f_3 is a basis of a three-dimensional oriented T-space, and $f_1 \equiv 1$, the problem is reduced to finding a curve in $\mathbb{R}^2$ such that for all $x \in \mathbb{R}^2$ there is a line through x which intersects the curve in two points.

In Examples 1, 2, and 3, Theorem 2 is not applicable because M does not have property (D). The point is that if M fails to have property (D), Lemma 4 need not hold, and, indeed, in all examples we have $Z_{n-1} \not\subset A$.

If M has property (D), but R is not oriented, Theorem 2 does not hold either, as is shown by the following example which is obtained from Example 2 by ordering M in a different way.

(4) Let $M = (0, \pi)$ and $R = \operatorname{span}\{f, g\}$ with

$$\begin{pmatrix} f \\ g \end{pmatrix}(t) = \begin{cases} \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} & \text{for} \quad t \in M \backslash \{1/n \mid n \in \mathbb{N}\}, \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{for} \quad t = 1, \\ \begin{pmatrix} \cos 1/(n-1) \\ \sin 1/(n-1) \end{pmatrix} & \text{for} \quad t = 1/n, \quad n = 2, 3, \ldots . \end{cases}$$

5. A Simple Proof of Mairhuber's Theorem

THEOREM 3 (Mairhuber). *Let M be a compact Hausdorff-space and $R \subset C(M)$ an n-dimensional T-space, and $n \geqslant 2$. Then M is homeomorphic to a topological subspace of the unit circle S^1.*

Proof. $n = 2$. If f, g is a basis of R, the function $h = M \to S^1$ defined by

$$h(t) = \left(\frac{f(t)}{(f^2(t) + g^2(t))^{1/2}}, \frac{g(t)}{(f^2(t) + g^2(t))^{1/2}} \right)$$

is a homeomorphism, for h is continuous and injective, and as M is a compact Hausdorff-space, h^{-1} is also continuous.

$n - 1 \Rightarrow n$. For every $z \in M$ we consider the restriction $E^M_{M\backslash\{z\}}(R)$ of R to $M\backslash\{z\}$. Because of Corollary 1 $E^M_{M\backslash\{z\}}(R)$ contains an $(n - 1)$-dimensional T-subspace. From the induction hypothesis we conclude that for every nonempty open subset U of M, the set $M\backslash U$ is homeomorphic to a subset of S^1. If for such a U the set $M\backslash U$ were homeomorphic to all of S^1, $E^M_{M\backslash U}(R)$ were an n-dimensional T-space over $M\backslash U$ containing an $(n - 1)$-dimensional T-space, which is impossible by Example 1 above. So for every nonempty open set $U \subset M$ the set $M\backslash U$ is homeomorphic to a proper subset of S^1, respectively, to a subset of $\mathbb{R}$. If M is not connected, i.e., there are two open, nonempty sets $A, B \subset M$ with $A \cap B = \varnothing$ and $A \cup B = M$, then $B = M\backslash A$ is homeomorphic to a proper closed subset of $\mathbb{R}$, and the same holds for $A = M\backslash B$ and so for M as well.

Now let M be connected. As M is compact, it contains a proper subset L which is connected and compact and is not a point (see, e.g., [3, p. 213]). So L is homeomorphic to a closed interval and contains an open curve with endpoints a and b. $M\backslash K$ is again homeomorphic to a subset of R. We distinguish two cases:

(1) If $M\backslash K$ is connected, it is a curve containing a and b. So M contains a closed curve $\mathscr{O}$. If $\mathscr{O}$ were not equal to M, $U := M\backslash\mathscr{O}$ were a nonempty set open in M, and $\mathscr{O}$ were homeomorphic to a proper subset of $\mathbb{R}$ and could not be a closed curve. So we have $\mathscr{O} = M$.

(2) If $M\backslash K$ is not connected, there are two nonempty closed sets $A, B \in M\backslash K$ with $A \cap B = \varnothing$ and $A \cup B = M\backslash K$. Let $a \in A$. If b were in A, too, $K \cup A$ would be a closed set. As $K \cup A$ and B are disjoint, and $(K \cup A) \cup B = M$, M would be disconnected. So we have $b \in B$. If A were disconnected, i.e., there were two nonempty closed subsets C and D of A with $C \cap D = \varnothing$ and $C \cap D = A$, let $a \in C$. Then $K \cup B \cup C$ would be a closed set. As $K \cup B \cup C$ and D are disjoint, and $(K \cup B \cup C) \cup D = M$, M would be disconnected. So A as well as B is connected. Therefore, A and B are simple curves closed in $\mathbb{R}$. Let $Y \subset K$ be an open curve with $a, b \notin \bar{Y}$. As $M\backslash Y$ is homeomorphic to a subset of $\mathbb{R}$, it contains no tripod-like set. So $M = A \cup K \cup B$ is a simple curve.

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